Journal of Applied Mechanics and Technical Physics, Vol. 40, No. 3, 1999

SELF-SIMILAR SOLUTIONS OF THERMAL TWO-PHASE FILTRATION

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UDC 517.958.532

This work is concerned with deriving generalized self-similar solutions for a thermal model of two-phase filtration in porous media.

The enormous increase in oil production all over the world has been made possible not only by an increase in the amount of proved reserves but also by the implementation of advanced mining methods that permit, in particular, to exploit oil pools with high-viscosity and paraffin oils, which cannot be extracted by conventional water flood methods.

Among the new mining methods for oil fields, different methods of thermal treatment of pools, for example, by oriented expulsion of oil by heat carriers (vapor or hot water) from injection to production wells or by cyclical hot-vapor treatment of wells are employed most widely. To describe this treatment, Monakhov and Bocharov [1, 2] were the first to propose a temperature model of two-phase filtration (the MLT model). It is based on the Muskat–Leverett isothermic model [3] and takes heat effects into account via the known dependences on the temperature, viscosity, and capillary properties of the two-phase liquid components.

In contrast to previous thermal models of two-phase filtration, the MLT model is, first, scientifically based in the sense that it uses only experimentally determined functional parameters, and, second, the equation of energy in this model is a consequence of the energy conservation laws for the components of the liquid and porous medium.

Self-similar solutions of the MLT model are constructed. In the isothermal case ($\theta \equiv \text{const}$), the existence of such solutions is established in [4, 5].

1. MLT Model. Let s_i, $\rho_i = \text{const}$, p_i and $\boldsymbol{v}_i$ ($i = 1, 2$) be the phase saturations (concentrations), densities, pressures, and velocities (flow rates) of filtration, and $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ be the velocity of filtration of the mixture. The equations of the MLT model have the form

$$m_0 \frac{\partial}{\partial \tau}(s_i \rho_i) + \nabla \cdot (\rho_i \boldsymbol{v}_i) = 0, \qquad \boldsymbol{v}_i = -K(\xi) \frac{k_i(s)}{\mu_i(\theta)}(\nabla p_i + \rho_i \boldsymbol{g}),$$

$$p_2 - p_1 = p_c(\xi, s, \theta), \qquad \frac{\partial \theta}{\partial \tau} = \nabla \cdot (\lambda \nabla \theta - \boldsymbol{v}\theta), \qquad s_1 + s_2 = 1.$$

Here m_0 is the porosity of the medium, θ is the temperature, K is the absolute permeability tensor for the medium, k_i and μ_i are the relative permeabilities and viscosity of the phases, $s = (s - s_1^0)(1 - s_1^0 - s_2^0)^{-1}$ is the effective saturation of the wetting phase, s_i^0 are the residual phase saturations, $\lambda = \sum\limits_{i=1}^{3} \alpha_i \lambda_i (\rho_i c_{pi})^{-1}$, $\alpha_i = m_0 s_i$ ($i = 1, 2$) and $\alpha_3 = 1 - m_0$ are the volume concentrations of the liquids and the porous medium, $\lambda_i = \lambda_i(\theta)$ and $c_{pi} = \text{const}$ ($i = 1, 2, 3$) are the phase thermal conductivities and heat capacities, $p_c(\xi, s, \theta)$ is the capillary pressure, τ is time, and $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ is the vector of space coordinates.

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 40, No. 3, pp. 9–17, May–June, 1999. Original article submitted August 27, 1998.

0021-8944/99/4003-0367 \$22.00

In the case of one-dimensional filtration of a two-phase liquid in a homogeneous porous medium in the plane orthogonal to the free-fall acceleration vector $\boldsymbol{g}$ (plan filtration), the equations of the MLT model become

$$\boldsymbol{u}_\tau = (A\boldsymbol{u}_\xi - \boldsymbol{B}q)_\xi, \qquad \boldsymbol{u} = (s, \theta). \tag{1}$$

Here $q = v(\tau + 1)^{-1/2}$ is the specified flow rate of the mixture ($v = \text{const}$), $A = A(\boldsymbol{u}) = \{a_{ij}\}$ is a square matrix, $\boldsymbol{B}(\boldsymbol{u}) = (b(\boldsymbol{u}), \theta)$, $a_{1i} = a(s)\alpha_{1i}$, $a(s) = Kk_1k_2$ ($K = \text{const}$), $\alpha_{11} = \nu(\boldsymbol{u})|p_{cs}|$, $\alpha_{21} = \nu|p_{c\theta}|$, $b = k_1\mu_2\nu$, $\nu^{-1} = (\mu_2 k_1 + \mu_1 k_2)m$, $m = m_0(1 - s_1^0 - s_2^0)$, $a_{12} = 0$, and $a_{22} = \lambda$.

We use the standard designations of Banach spaces of Hölder continuous functions $f(x) \in C^\alpha(\bar\Omega)$ [$\alpha > 0$ and $x = (x_1, \dots, x_n)$] with a norm $\|f\|^{(\alpha)} = \max\limits_{(x,y)\in\Omega}(|f(x)| + |f(x) - f(y)||x - y|^{-\alpha})$, continuously differentiable functions $f(x) \in C^1(\bar\Omega)$ with a norm $\|f\|^{(1)} = \max(|f(x)| + |\nabla f(x)|)$, and functions $f(x) \in C^{1+\alpha}(\bar\Omega)$ with a norm $\|f\|^{(1+\alpha)} = \|f\|^{(\alpha)} + \|\nabla f\|^{(\alpha)}$.

According to the properties of the functional parameters of the MLT model, the coefficients of (1) $(A, \boldsymbol{B}) \in C^\alpha(R)$, $R = [0,1] \times [\theta_0, \theta_1]$, and satisfy the conditions [1]

$$\begin{gathered} M_0^{-1} \leqslant (a^{-1}a_{11}, a_{22}) \leqslant M_0, \qquad (a^{-1}|a_{12}|, a^{-1}|b_\theta|, |b_s|, |a_s|) \leqslant M_0, \\ a_{21} = 0, \qquad 0 < a(s) < 1, \quad s \in (0,1), \quad a(0) = a(1) = 0. \end{gathered} \tag{2}$$

Let filtration of the two-phase liquid occur between two wells (galleries of wells) located on the lines $\xi = \xi_i$ ($i = 0, 1$). For definiteness, the line $\xi = \xi_0 = 0$ corresponds to an injection well, and $\xi = \xi_1$ to a production well. At the wells, we specify either the flow rate of the wetting phase (water)

$$-v_1\Big|_{\xi=\xi_k} = (a_{11}s_\xi + a_{12}\theta_\xi - vb)_{\xi=\xi_k} = -(vb)\Big|_{\xi=\xi_k},$$

which is proportional to the phase mobility (the right side of the condition), or the saturation of the wetting phase $s\Big|_{\xi=\xi_k} = s_k$. Similarly, at $\xi = \xi_k$, the heat flux $\sigma = \lambda\theta_\xi - v\theta$ or temperature is specified.

Thus, the boundary and initial conditions for system (1) take one of the following forms:

$$\boldsymbol{u}\Big|_{\xi=\xi_k} = \boldsymbol{u}_k(\tau), \qquad \boldsymbol{u}\Big|_{\tau=0} = \boldsymbol{u}^0(\xi), \quad k = 0, 1; \tag{3}$$

$$(v_1 - vb, \sigma - \sigma_k)\Big|_{\xi=\xi_k} = 0, \qquad \boldsymbol{u}\Big|_{\tau=0} = \boldsymbol{u}^0(\xi), \quad k = 0, 1. \tag{4}$$

2. Self-Similar Variables. Regularization. The variables $t = \ln(1 + \tau)$ and $x = \xi(\tau + 1)^{-1/2}$, called self-similar, allow system (1) to be written as

$$\boldsymbol{u}_t = (A\boldsymbol{u}_x - \boldsymbol{B}v)_x + \frac{1}{2}x\boldsymbol{u}_x \equiv L\boldsymbol{u}. \tag{5}$$

Definition 1. Steady ($\boldsymbol{u}_t = 0$) solutions of the boundary-value problem for Eq. (5)

$$L\boldsymbol{u} \equiv (A\boldsymbol{u}_x - \boldsymbol{B}v)_x + \frac{1}{2}x\boldsymbol{u}_x = 0, \qquad \boldsymbol{u}\Big|_{x=x_k} = \boldsymbol{u}_k, \quad k = 0, 1, \tag{6}$$

where $x \in \Omega = \{x | 0 = x_0 < x < x_1 \leqslant \infty\}$, are called *self-similar solutions* of problem (1), (3) (of a parabolic type).

Self-similar solutions of (6) allow one, in particular, to obtain numerical solutions of the initial equations (1).

Since Eq. (6) is homogenous in θ with respect to derivatives, the linear substitution $\theta = \gamma_1 T + \gamma_0$ ($\gamma_i = \text{const}$) brings boundary conditions (6) to the form $T(x_0) = 1$ and $T(x_1) = 0$. Therefore, without loss of generality, in (6) we set $\theta_0 = 1$, $\theta_1 = 0$, and $s_k \in [0,1]$.

Without changing the notation, in (6) we set $a_{11} = a_\varepsilon\alpha_{11}$ and $a_\varepsilon = a(s) + \varepsilon$, where $\varepsilon > 0$. We continue A and $\boldsymbol{B}$ outside the intervals $s \in [0,1]$ and $\theta \in [0,1]$ by their extreme values and perform Steklov averaging of the coefficients α_{11}, α_{12}, and $a_{22} = \lambda$ of the matrix A: $\alpha_{11}(h)$, $\alpha_{12}(h)$, and $\lambda(h)$, where $h \to 0$ is the averaging parameter.

Along with the form (6) of the problem regularized by the method described above, we consider its equivalent representation

$$a_2u_{xx} + a_1u_x + af = 0, \qquad \lambda\theta_{xx} + \lambda_1\theta_x = 0, \qquad v(x_k) = v_k, \qquad k = 0, 1. \tag{7}$$

Here $v = (u, \theta)$, $u = \int\limits_0^s a_\varepsilon(t)\,dt$, $a_2 = \alpha_{11}(h)$, $a_1 = \alpha_{11x}(h) + a_sa_\varepsilon^{-1}\alpha_{12}(h)\bar\theta_x - b_sa_\varepsilon^{-1}v + 0.5xa_\varepsilon^{-1}$, $f = \alpha_{12x}(h)\bar\theta_x - \alpha_{12}(h)\lambda_1\lambda^{-1}\bar\theta_x - b_\theta a^{-1}v\bar\theta_x$, $\lambda_1 = \lambda_x - v + 0.5x$, $\lambda = a_{22}(h)$, and $\bar\theta_x$ is the cut-off of θ_x: $\bar\theta_x = \theta_x$ for $|\theta_x| \leqslant M_1$ and $|\bar\theta_x| = M_1$ for $|\theta_x| > M_1$ ($M_1 = \text{const} > 0$ is fixed below). With this regularization, the coefficients of system (7) are bounded: $(|a_2|, a_1, \lambda, |\lambda_1|, |f|) \leqslant \bar M(M, h)$.

3. Solvability of the Regularized Problem. Lemma 1 (on estimates). *For solutions $v = (u, \theta)$ of problem* (7), *the following estimates hold:*

$$0 \leqslant \theta(x) \leqslant 1, \qquad |\theta_x| \leqslant M_1e^{-\alpha x^2}; \tag{8}$$

$$0 \leqslant u(x) \leqslant u_0, \qquad \alpha_0a|s_x| \leqslant |u_x| \leqslant M_2(x_1). \tag{9}$$

Here the constants M_1 and M_2 are independent of ε and h.

Proof of the first estimate (8) begins with the introduction of the new function $\omega(x)$: $\theta = (\gamma - e^{-\beta x})\omega(x) \equiv \alpha\omega$ which satisfies the equation

$$L_0\omega = \alpha\lambda\omega_{xx} + (\lambda_1\alpha + 2\beta\lambda e^{-\beta x})\omega_x - \beta e^{-\beta x}(\lambda\beta - \lambda_1)\omega = 0.$$

We choose $\beta > 0$ so large that $(\lambda\beta - \lambda_1) > 0$, $x \in \Omega \equiv [0, x_1]$ and fix $\gamma > 1$, so that $\alpha = \gamma - e^{-\beta x} > 0$ ($x \in \Omega$). Let there exist, at a certain point $x_2 \in \Omega$, a negative minimum $\omega(x)$, $\omega(x_2) < 0$. Then, $\omega_x(x_2) = 0$, $\omega_{xx}(x_2) \geqslant 0$, and $L_0\omega(x_2) \geqslant \beta e^{-\beta x_2}(\lambda\beta - \lambda_1)|\omega(x_2)| > 0$, which contradicts the equality $L_0\omega = 0$. Consequently, $\omega(x) \geqslant 0$ and, hence, $\theta(x) \geqslant 0$ ($x \in \bar\Omega$). The inequality $\bar\theta \equiv 1 - \theta \geqslant 0$ is obtained in the same manner.

We write the boundary-value problem (6) for the function $\theta(x)$ in the form

$$(\lambda\theta_x)_x + \lambda_0(\lambda\theta_x) = 0, \qquad \theta(x_0) = 1, \qquad \theta(x_1) = 0,$$

where $\lambda_0 = (0.5x - v)\lambda^{-1}$. Regarding $\lambda(x) \equiv \lambda[s(x), \theta(x), h]$ and $\lambda_0(x) = \lambda_0[s(x), \theta(x), h]$ as specified functions, we come to the following representation of solutions of the last problem:

$$1 - \theta = NF(x), \qquad F = \int\limits_0^x \lambda^{-1}(t)e^{-\Lambda(t)}\,dt, \qquad \Lambda = \int\limits_0^x \lambda_0(t)\,dt, \tag{10}$$

where $N = [F(x_1)]^{-1}$.

To extend the consideration to the case $x_1 = \infty$, which is typical of mechanical problems in self-similar variables, we write the following inequalities, which are consequences of (10) and lead to the second estimate (8) for θ_x:

$$\gamma_0^{-1}e^{-\alpha x^2} \leqslant \frac{dF}{dx} \leqslant \gamma_0e^{-\alpha x^2}, \qquad \gamma_0 = \max(\lambda, \lambda^{-1}),$$

The first estimate of (9) is obtained in the same manner as (8) for $\theta(x)$ by the substitution

$$u(x) = (\gamma - e^{-\beta x})w(x) \equiv \alpha w, \qquad \gamma > 1, \qquad a_{11}\beta - a_1 > 0,$$

which reduces (7) to the following equation for $w(x)$:

$$L_1w \equiv a_2w_{xx} + a_3w_x - cw + af = 0, \qquad x \in \Omega = (0, x_1) \quad (c > 0).$$

Let, at a point $x_2 \in \Omega$, a negative minimum of the function $w(x)$ be reached, i.e., $w(x_2) < 0$. By virtue of regularization, $a(s) = 0$ for $s \notin [0, 1]$ $[u(x_2) = \alpha(x_2)w(x_2) < 0]$, and then, $L_1w(x_2) \geqslant c|w(x_2)| > 0$, which is inconsistent with the validity of the equation $L_1w(x_2) = 0$ ($x_2 \in \Omega$) and, hence, $u(x) = \alpha w \geqslant 0$ ($x \in \bar\Omega$). Similarly, introducing the function $z = (u_0 - u)(\gamma - e^{-\beta x})^{-1}$, we obtain the upper bound $u(x) \leqslant u_0$.

The estimate $|\theta_x|$ allows us to eliminate the cutoff $\bar{\theta}_x$ in the coefficients a_1 and f and write problem (7) for $u(x) = \int\limits_0^s a_\varepsilon(t)\,dt$ in the form

$$(a_2 u_x + \varphi)_x = 0, \qquad u(0) = u_0, \qquad u(x_1) = u_1.$$

Integration of this problem yields

$$-a_2 u_x = \varphi + C, \qquad u(0) = u_0 \qquad (C = \text{const}). \tag{11}$$

Here $\varphi = 0.5\Big[xs + \int\limits_x^{x_1} s(t)\,dt\Big] + a_{12}\theta_x - vb$, $u_0 = \int\limits_0^1 a_\varepsilon(t)\,dt$, $C = C_0^{-1}\Big[u_0 - u_1 - \int\limits_0^{x_1} a_2^{-1}(t)\varphi(t)\,dt\Big]$, and $C_0 = \int\limits_0^{x_1} a_2^{-1}(t)\,dt$.

Representation (11) obviously leads to the second estimate (9). Lemma 1 is proved.

Remark 1. Equations (7) [the equation for $u(x)$ is multiplied by $a_\varepsilon(s)$] lead directly to the following estimates:

$$a|(as_x)_x| \leqslant M_2(x_1), \qquad |(\lambda\theta_x)_x| \leqslant M_1 e^{-\alpha x^2}.$$

Lemma 2 (on Hölder continuity). *Let*

$$a(s) \geqslant a_0 s^{\alpha_0}(1-s)^{\alpha_1}, \qquad a_0 = \text{const} > 0 \qquad (\alpha_0, \alpha_1 \geqslant 0). \tag{12}$$

Then for solutions $\boldsymbol{u}(x) = (s,\theta)$ *of problem* (6) *the following inequalities hold:*

$$(\|s\|^{(\beta)}, \|a_\varepsilon s_x\|^{(\beta)}, \|\theta_x\|^{(\beta)}) \leqslant M_3(x_1), \qquad \beta = (1+\alpha)^{-1}, \qquad \alpha = \max(\alpha_0, \alpha_1). \tag{13}$$

Proof. We first establish the Hölder continuity of the transformation $s = s(u)$, which is inverse to $u = \int\limits_0^s a_\varepsilon(t)\,dt$ $(a_\varepsilon = a + \varepsilon \geqslant a)$:

$$|s(u_2) - s(u_1)| \leqslant K|u_2 - u_1|^\beta, \qquad (u_1, u_2) \in [0, p], \qquad p = \int\limits_0^1 a_\varepsilon(t)\,dt.$$

For this, it apparently suffices that $|u(s_2) - u(s_1)| \geqslant K_0|s_2 - s_1|^{1+\alpha}$ for $(s_1, s_2) \in [0.3/4] \cup [1/4.1]$.

For definiteness, let $0 \leqslant s_1 \leqslant s_2 \leqslant 3/4$. Then, $u_2 - u_1 = \int\limits_{s_1}^{s_2} a_\varepsilon(s)\,ds \geqslant K_0 \int\limits_{s_1}^{s_2} s^\alpha(1-s)^\alpha\,ds \geqslant K(s_2^{\alpha+1} - s_1^{\alpha+1}) \geqslant K(s_2 - s_1)^{\alpha+1}$, $K = K_0 4^{-\alpha}(1+\alpha)^{-1}$. The last of the chain of inequalities follows from the consideration of the function $f(\sigma) = (1-\sigma^\gamma)(1-\sigma)^{-\gamma}$ $(\gamma = 1+\alpha$ and $\sigma = s_1/s_2)$, for which $\min f(\sigma) = f(0) = 1(f_\sigma > 0, 0 < \sigma < 1)$.

Thus, it is proved that $s(u) \in C^\beta[0,p]$. Since $|u_x| \leqslant M_2$, we have $s[u(x)] \in C^\beta[0, x_1]$. Then, representations (10) and (11), in which the coefficients $\lambda(x) \equiv \lambda[s(x), \theta(x)]$, $a_2(x)$, etc. are continuous after Hölder, lead to inequalities (13). Lemma 2 is proved.

Lemma 3 (on the solvability of the regularized problem). *Let* $(A, B) \in C^\alpha(R)$, $R = (0,1)\times(0,1)$, *and conditions* (2) *be satisfied. Then, the regularized problem* (6) $\forall(\varepsilon, h) > 0$ *has at least one classical solution* $\boldsymbol{u} = (s,\theta)$, *for which estimates* (8) *and* (9) *are valid. Under the additional condition* (12), $u(x)$ *satisfies inequalities* (13).

Proof. We write problem (6) for $\boldsymbol{u} = (s,\theta)$ in equivalent form:

$$\frac{d^2\boldsymbol{u}}{dx^2} = \boldsymbol{F}(x, \boldsymbol{u}, \boldsymbol{u}_x), \qquad \boldsymbol{u}(x_k) = \boldsymbol{u}_k, \qquad k = 0, 1.$$

By virtue of the regularization of the coefficients (6), estimates (8) and (9), and their consequence $|s_x| \leqslant$

$a_\varepsilon^{-1}|u_x| \leqslant M_4(x_1,\varepsilon)$, we have $|\boldsymbol{F}| \leqslant M_5(x_1,\varepsilon)$ for $(s,\theta) \in [0,1]$ and $(s_x,\theta_x) \in (-\infty,\infty)$ $\forall(\varepsilon,h) > 0$. The solvability of the above problem and problem (6) follows from the Birkhoff–Kellog theorem [6]. Lemma 3 is proved.

4. Generalized Solutions. Definition 2. A vector $\boldsymbol{u} \in C^\alpha(\bar{\Omega})$ which satisfies the boundary conditions, inequalities (8), (9), and (13), and the integral identity

$$\int_0^{x_1} \left[(A\boldsymbol{u}_x - \boldsymbol{B}v)\eta_x + \frac{1}{2}(x\eta)_x\boldsymbol{u}\right] dx = 0 \tag{14}$$

$\forall\eta \in C^1(\bar{\Omega})$, $\eta(0) = \eta(x_1) = 0$ is called a *generalized solution* of problem (6).

Remark 2. By virtue of (8), (9), and (13), we have $(u,\theta) \equiv \boldsymbol{v} \in C^{1+\alpha}(\bar{\Omega})$ ($\alpha > 0$ and $u = \int_0^s a_\varepsilon(t)\,dt$), and, hence, constructing a generalized solution of problem (6) is equivalent to solving the Cauchy problem

$$-a_2u_x = \varphi + C, \qquad -\lambda\theta_x = \psi + K, \qquad \boldsymbol{v}(0) = \boldsymbol{v}_0. \tag{15}$$

Here the function φ and the constant C are defined in (11), $\psi = 0.5\Big(x\theta + \int_x^{x_1}\theta(t)\,dt\Big) - v\theta$, $K = K_0^{-1}\Big(1 - \int_0^{x_1}\psi(t)\lambda^{-1}(t)\,dt\Big)$, and $K_0 = \int_0^{x_1}\lambda^{-1}(t)\,dt$.

Existence Theorem 1. *Let* $(A,\boldsymbol{B}) \in C^\alpha(\bar{R})$, $\alpha > 0$, $R = (0,1)\times(0,1)$, and (2) and (12) hold. *Then, there is at least one generalized solution of problem* (6).

Proof. From the sequence $\{u(x,\varepsilon,h),\theta(x,\varepsilon,h)\} \in C^{1+\alpha}(\bar{\Omega})$, $u = \int_0^s a_\varepsilon(t)\,dt$ we separate out the subsequence $\{u(x,\varepsilon_k,h_k),\theta(x,\varepsilon_k,h_k)\}$, which converges at $C^{1+\alpha_0}(\bar{\Omega})$, $0 < \alpha_0 < \alpha$ as $(\varepsilon_k,h_k) \to 0$. Passing to the limit $(\varepsilon_k,h_k) \to 0$ in the integral identity (14), we obtain a generalized solution $\{u(x),\theta(x)\}$ of problem (5).

Theorem 2 (on the finite velocity). *Let, in addition to the conditions of Theorem* 1, *the following inequalities hold*:

$$(a(s),b(s,\theta)) \leqslant Ms^\gamma, \quad \gamma \geqslant 1 \quad \{or\,(a,|b-b(1,0)|) \leqslant M(1-s)^\gamma\}. \tag{16}$$

Then, for $s(x_1) = 0$ $\{or\ s(x_1) = 1\}$, $x_1 \gg 1$, *there is a value* $x_* < \infty$ *such that*

$$s(x) \equiv 0 \quad \{or\ s(x) \equiv 1\} \quad at\ x \geqslant x_*, \tag{17}$$

i.e., the front $s = 0$ $(s = 1)$ *propagates at finite velocity.*

Proof. We set

$$x_* = (X^2 + M_0\delta^{-1})^{1/2}, \qquad X = 2\max(|a_{12}|s^{-1}M_1,\ |v|b_*s^{-1}), \tag{18}$$

where $M_0 = \int_0^1 a(t)t^{-1}\,dt$, $\delta = (1/8)\min\alpha_{11}$, $M_1 = \max|\Theta_x|$, $b_* = b$ for $s(x_1) = 0$ and $b_* = |b - b(1,0)|$ for $s(x_1) = 1$.

We consider representation (11), (15) for $u(x)$. Since $u(x_1) = 0$ and $u \geqslant 0$ in the neighborhood $x = x_1$, we apparently have $u_x(x_1) \leqslant 0$ and, hence, $c = -a_2(0,0)u_x(x_1) \geqslant 0$. Then, $-a_2u_x = \varphi + c \geqslant 0.5xs - |a_{12}||\theta_x| - |v|b \geqslant xs/4$ at $x \geqslant X$, where X is defined by (18). Thus, we come to the inequality

$$[u(s)]_x + 2\delta xs \leqslant 0, \quad \delta = (1/8)\min\alpha_{11},$$

which is equivalent to

$$[\Phi(s)]_x + 2\delta x \leqslant 0, \qquad x \in [X,x_1], \qquad s(X) = s_2 \geqslant 0, \tag{19}$$

where $\Phi(s) = \int\limits_0^s a(t)t^{-1}\,dt$. From (19) it follows that $\Phi_x < 0$ at $x \geqslant X$, and $\Phi_s \geqslant 0$ ($s \in [0,1]$). Therefore, $s(x) \leqslant s(y)$ at $x \geqslant y$. Integrating (19), we obtain

$$-\int\limits_s^{s_2} a(t)t^{-1}\,dt + \delta(x^2 - X^2) \leqslant 0 \qquad (x \geqslant X).$$

Since

$$\Phi(s) - \Phi(s_1) = -\int\limits_s^{s_2} a(t)t^{-1}\,dt \geqslant -\int\limits_0^1 a(t)t^{-1}\,dt \equiv -M_0,$$

from the previous inequality it follows that $\delta(x^2 - X^2) - M_0 \leqslant 0$ ($x \geqslant X$), which is possible only at $X \leqslant x \leqslant x_* \equiv (X^2 + M_0\delta^{-1})^{1/2}$.

In order that (19) be valid for $x > x_*$, relation (17) must be satisfied.

If the problem $s(0) = s_0 \geqslant 0$, $s(x_1) = 1$ is solved, then substituting $\sigma = 1 - s$ and assuming that $b_* = |b(s,\theta) - b(1,0)|$, we come to the above case $\sigma(x_1) = 0$, which leads to the identity $\sigma = 1 - s \equiv 0$ at $x \geqslant x_*$. Theorem 2 is proved.

Remark 3. Let the boundary-value problem for system (6) be solved in the interval $[-x_0, x_1]$, $(x_0, x_1) \gg 1$, and $s(-x_0) = 1$ and $s(x_1) = 0$. Then, according to Theorem 2, we have a solution of the type of a surf: $s(x) \equiv 1$ at $x \leqslant -x_*$ and $s(x) \equiv 0$ at $x \geqslant x_*$

Remark 4. Estimates (8) for $\theta(x)$ and the proved finite velocity for $s(x)$ [relations (17)] obviously lead to the existence of a generalized solution of problem (6) at $x_1 = \infty$.

5. Mixed Boundary-Value Problem. For system (6), we consider the following boundary-value problem of type (4):

$$(A\boldsymbol{u}_x - v\boldsymbol{B})(x_0) = \boldsymbol{Q}, \qquad \boldsymbol{u}(x_1) = \boldsymbol{u}_1, \qquad v \geqslant 0. \tag{20}$$

Here $x_0 = 0, l$ and $x_1 = l, 0$, where $l > 0$ is finite or $l = \infty$; the vector $\boldsymbol{Q} = (-Q, q)$ is specified (Q is the flow rate of the wetting phase and q is the heat flux).

We study the cases of physically feasible conditions on the parameters of the boundary-value problem (20). In this case, for solutions of problem (6), (20) [system (6) and boundary conditions (20)], Theorems 1 and 2, proved for the first boundary-value problem (6), are valid.

We first consider the case $x_0 = 0$, $x_1 = l \gg 1$. Integrating (20), for $\theta(x)$ we obtain

$$\lambda\theta_x = N\mathrm{e}^{-\Lambda(x)}, \qquad \Lambda = \int\limits_0^x \lambda^{-1}(t)(t/2 - v)\,dt, \tag{21}$$

where N is the required constant. Relation (21) leads to the inequality $|\Lambda|x^{-2} \leqslant M_0$, $x \geqslant 1$. Substituting (21) into (20), we obtain

$$(\lambda\theta_x - v\theta)(0) = N + Nv\int\limits_0^l \lambda^{-1}(t)\mathrm{e}^{-\Lambda(t)}\,dt - v\theta_1 = q,$$

and, hence, $N = (q + v\theta_1)\Big(1 + v\int\limits_0^l \lambda^{-1}\mathrm{e}^{-\Lambda}\,dt\Big)^{-1}$, $0 < \delta_0 \leqslant N \leqslant \delta_0^{-1}$. Then

$$\theta(0) = \theta_1 - N\int\limits_0^l \lambda^{-1}\mathrm{e}^{-\Lambda}\,dt \geqslant \theta_1 - (q + v\theta_1)v \equiv \theta_*,$$

i.e., $\theta_* < \theta^* = \theta_1 = \theta(l)$. Assuming formally that θ_* is specified, we come to the first boundary-value problem

(6) studied above, and from Lemma 1 we find that

$$\theta_* \leqslant \theta(x) \leqslant \theta_1 \equiv \theta^*, \qquad |\theta_x| \leqslant M_1 e^{-\alpha x^2}. \tag{22}$$

Because estimates (22) do not depend on l, we can set $l = \infty$.

We turn to problem (6), (20) for water saturation $s = s(x)$. If at the point $x = 0$ at the well, the flow rate of the liquid is proportional to its mobility ($Q = vb\big|_{x=0}$) and $s(x_1) = s_1 = 0$, then, from the equality $b\big|_{s=0} = 0$ we find that $s \equiv 0$ is a solution of problem (6), (20). If $s(x_1) = 1$, we substitute $\sigma = 1 - s$ and then, as above, $\sigma \equiv 0$ is a solution of the converted problem (6), (20).

Thus, for $Q = vb\big|_{x=0}$, it makes sense to study problem (6), (20) only if $s(x_1) \neq 0, 1$.

We consider problem (20) for the regularized equation (7) for $u = \int\limits_0^s a_\varepsilon(t)\,dt$. The coefficients of this equation are bounded and $af\big|_{u=0,u_1} = 0$, where $u_1 = \int\limits_0^1 a_\varepsilon(t)\,dt$. Thus, the conditions of Lemma 1, according to which $0 \leqslant u(x) \leqslant u_1$ and $|u_x| \leqslant M_2(x_1)$, are satisfied, and these estimates ensure solvability of problem (7), (20) (Lemma 3 and Theorem 1) for $x_1 < \infty$. When the additional conditions (16) are satisfied, Theorem 2 on the finite velocity is valid, and, hence, problem (6), (20) is solvable at $x_1 = \infty$ as well.

We now consider the case of problem (20) where $x_0 = l \to \infty$ and $x_1 = 0$. This circumstance does not introduce additional difficulties in determining the function $\theta(x)$ compared to the case $x_0 = 0$ and $x_1 \to \infty$.

We assume that there is $u_\infty = \lim\limits_{x\to\infty} u(x)$, and $(|u'|, |u''|) \leqslant M$, $x \in [0, \infty)$. Then, $u'(x) \to 0$ as $x \to \infty$ [7, p. 200].

Since, by virtue of (22), $\theta_x \to 0$ as $x \to \infty$, conditions (20) at the point $x_0 = \infty$ take the form

$$\theta_\infty = -qv^{-1}, \qquad b(s_\infty, \theta_\infty) = Qv^{-1}. \tag{23}$$

By virtue of the uniqueness of $b(s, \theta)$ for s ($b_s \neq 0$, $s \neq 0, 1$) the values of s_∞ and θ_∞ are uniquely determined from relations (23).

In particular, $s_\infty = 1$ for $\theta = v$ and $s_\infty = 0$ for $v = 0$, and θ_∞ is not determined. Thus, for $x_0 = \infty$ and $x_1 = 0$ problem (6), (20) is equivalent to the one studied previously $(s, \theta)(0) = (s_1, \theta_1)$, $(s, \theta)(\infty) = (s_\infty, \theta_\infty)$, where s_∞ and θ_∞ are uniquely determined from (23).

Remark 5. If the flow rate at the well $x_0 = \infty$ is proportional to the mobilities of the phases, i.e., $Q = vb|_{x=\infty}$, conditions (23) do not determine the values of s_∞.

6. Numerical Implementation. Zhumagulov et al. [8] describe the algorithmic fundamentals of the project "New Computer Technologies in Oil Production" developed by the authors. One of the central points of the project is a program for the numerical calculation of self-similar solutions of the MLT model and solutions on of the unsteady problem (5) in self-similar variables.

It is known that the self-similar variables $t = \ln(1 + \tau)$ and $x = \xi(1 + \tau)^{-1/2}$ adequately describe the motion of oil–water contact, the dynamics of the regions of the well effect, etc. At the same time, the dynamics of the main output characteristics of oil extraction is still evaluated on the basis of self-similar solutions of two-phase filtration. Therefore, the problem of numerical construction of self-similar solutions of the MLT model has assumed an applied significance. However, numerical implementation of self-similar solutions is hampered by several circumstances:

(1) infiniteness of the interval of integration ($x \in [0, \infty)$),

(2) non-evolution nature of boundary conditions (6), (20),

(3) degeneration of Eq. (6) for $s(x)$ ($a_{11}\big|_{s=0,\,1} = 0$),

and others.

These features of problem (6) were taken into account in the development of numerical algorithms.

1. To obtain a solution of problem (6) in a finite interval, we used the theoretical estimates (18) of the front $x = x_*$ of propagation of the perturbation for $s(x)$ and estimate (8) of the rate of convergence $\theta(x)$ as

$x \to \infty$. In this case, the length of the interval of integration $l < \infty$ was fixed.

2. To reduce problem (6) to an evolution problem, we used the following iteration of the boundary conditions by Cauchy data:

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}'(0) = -(\tan\alpha_1^{(n)}, \tan\alpha_2^{(n)}), \quad \alpha_i \in (0, \pi/2).$$

The process of iteration began with an arbitrary value $(\alpha_1^{(0)}, \alpha_2^{(0)})$. According to this, the value $s^{(0)}(l)$ or $\theta^{(0)}(l)$ was larger or smaller than zero, and $\alpha_k^{(1)} = 2\alpha_k^{(0)}$ or $\alpha_k^{(1)} = \alpha_k^{(0)}/2$ was used as $\alpha_k^{(1)}$. The convergence of such process in the isothermal case ($\theta \equiv \text{const}$) was established by Kazhikhov [5].

3. The algorithm of reducing Eq. (6) for $s(x)$ to a nondegenerate equation consists not only of its regularization — the substitution of $\tilde{a}_{11} = a_\varepsilon \alpha_{11}$ ($a_\varepsilon = a + \varepsilon$) for $a_{11} = a\alpha_{11}$ but also additional calculations of the function $u(x) = \int\limits_0^s a_\varepsilon(t)\,dt$ and subsequent determination of $s(x)$.

Several difference schemes of solving problem (6), which models the displacement of oil by hot water, were compared.

It is established numerically that heating of a pool increases the degree of washing of the pool considerably. The above algorithms of solution of problem (6) allow one, in particular, to rapidly evaluate the effectiveness of thermal methods of secondary exploitation of oil fields.

The present approaches are implemented in [9].

This work was supported by the Russian Foundation for Fundamental Research and the Program "The Universities of Russia."

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